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# Dynamical stability revisited

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**Abstract**
States are considered as dynamically stable if they are invariant under a time automorphism and depend smoothly on a perturbation of the dynamics. We study the consequences for finite systems and compare it with the consequences in infinite systems. With an appropriate definition of smoothness it is shown that states that are dynamically stable are equilibrium states. In contrast to previous results we do not need strong asymptotic Abelianess.

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## 1. Introduction

For finite systems equilibrium states are given as Gibbs states with the density matrix $\rho = \mathrm{e}^{-\beta H}/\mathrm{Tr}\,\mathrm{e}^{-\beta H}$ where the operator $H$ implements the time evolution. In the thermodynamic limit neither the Hamiltonian converges to an observable nor can the state be described by a density matrix. What remains is the analyticity property of the time correlation functions expressed as the KMS condition referring to the inverse temperature $\beta$ [1–3]. States that satisfy this KMS condition are therefore considered to be equilibrium states. In [4] dynamical stability was introduced as a concept that should explain what singles out equilibrium states from other time invariant states without referring to thermodynamical ideas as temperature or entropy and how in the thermodynamic limit such equilibrium states are just the KMS states, characterized by a temperature. In this concept they demanded that equilibrium states on the global i.e. macroscopic size be time invariant, but in addition should not be effected by small local perturbations of the dynamics in the sense that for a sufficiently small perturbation there exists a unique state invariant under the perturbed dynamics and close in norm to the original state, so that the system can react on the local perturbation without being effected globally. With the additional assumption, that the time evolution is norm asymptotically Abelian [4] succeeded to prove that this suffices that the state has to be a KMS state for some temperatures. Assuming asymptotic Abelianess was justified by the observation that it holds for free particles (in the case of fermions only for the even subalgebra) and the hope that interactions should not be able to create correlations over large times that would dominate the spreading of the wavepackets over large times. There exists hardly any time evolution for which we can control the long-time behaviour apart from quasifree evolutions, with the exception of the $XY$-model. Here we can find the exact behaviour for all times [5, 6], since by a Jordan–Wigner transformation [7] the time evolution coincides with a quasifree evolution on the even algebra and can be extended to the whole algebra. As a consequence, on the even algebra the time evolution is again asymptotically Abelian, however for the extension this fails. Since the Hamiltonian is in no sense exceptional this is a strong indication that hoping for strong or even norm asymptotic Abelianess in realistic models is too optimistic.

A little bit later than [4], in [8] another concept was used to describe equilibrium states. In [8] the states have been considered that are passive in the following sense: as a perturbation of time evolution is slowly switched on, the state is disturbed only microscopically, and when the perturbation is switched off again, the system has gained energy. This assumption is in correspondence to the second law of thermodynamics. Therefore, contrary to the concept of dynamical stability, passivity does not contribute to our understanding of why thermodynamic holds for macroscopic systems. It merely relates thermodynamic behaviour with equilibrium. The fact that we are dealing with infinite systems can be taken care of by considering space translations which commute with time translation, and the invariance of the state under space translations. No further assumptions on the dynamics were necessary. This was sufficient to prove that passive states are KMS states thereby showing that the concept of temperature is a consequence of the second law of thermodynamics.

Here we want to concentrate on dynamical stability but apply the methods which have formally been applied in the context of passivity. We assume that the state is dynamically stable and first examine the consequences for a finite system: the density matrix corresponding to the stable state has to commute with the Hamiltonian. We assume that we switch on the perturbation adiabatically. If the perturbation is sufficiently small there is no need to take care of level crossing, every eigenvector of the density matrix will evolve separately so that the weight of the subspaces will be unaltered. We obtain a unique perturbed state if the degeneracy of the initial density matrix coincides with the degeneracy of the Hamiltonian. Thus the density matrix has to be a function of the Hamiltonian. Passivity for finite-dimensional systems works similarly but reduces the permitted functions to convex functions. For an infinite system we have to take into account that the difference between the eigenvalues of the Hamiltonian will tend to zero in the thermodynamic limit. Therefore adiabatic perturbation of the eigenvectors cannot be applied. In [9] adiabatic theory for the scattering operator replaced it, but the existence of a scattering operator again relies on the assumption of asymptotic Abelianess which we now do not expect to be available. Nevertheless dynamical stability corresponds to that we assume we stay in the same representation, since the system is not effected globally by the perturbation, and that in correspondence to adiabatic perturbation theory the vector implementing the invariant state changes smoothly and uniquely. To first order we find a unique relation between perturbation of the dynamics and perturbation of the vector that implements the invariant state in the GNS construction. Mimicking ideas of [8] we will add invariance with respect to space translation to the list of assumptions. For passive states this invariance reduced the possible convex functions of the Hamiltonian to the KMS condition with positive temperature. Also in our situation the possible functions of the finite-dimensional situation are again reduced to KMS states, however without restrictions on the sign of the temperature. In fact, this cannot be expected since we have not specified a time direction.

In the appendix we give the arguments, why the time evolution of the $XY$-model is not asymptotically Abelian.

## 2. Dynamical stability for finite-dimensional algebras

We start with an algebra $\mathcal{A}$ isomorphic to $M_n$ and acting on a Hilbert space of dimension $n$. Its time evolution $\tau_t$ is given by a Hamiltonian $H = \sum_i^n h_i|\phi_i\rangle\langle\phi_i|$ such that $\tau_t(A) = \mathrm{e}^{\mathrm{i}Ht} A\, \mathrm{e}^{-\mathrm{i}Ht}$.

**Definition 1.** *A state $\omega(A) = \mathrm{Tr}\, \rho A$ is called dynamically stable with respect to $\tau_t$ if it satisfies the following conditions:*

*(i) The state is invariant under the dynamics $\tau_t$.*
*(ii) There exists some $\lambda_0 \in R^+$ such that $\forall\lambda$, $|\lambda| < \lambda_0$ and all $A = A^* \in \mathcal{A}$ with $\|A\| = 1$ there exists a unique state $\omega_{\lambda A}$ unitarily equivalent to $\omega$ that is invariant under the perturbed dynamics $\tau_t^{\lambda A}$ induced by $\mathrm{e}^{\mathrm{i}t(H+\lambda A)}$.*

**Remark.** We had to introduce the assumption that the states are unitarily equivalent to guarantee uniqueness. From a physical point of view we can justify this assumption, because it holds automatically if the perturbation is switched on adiabatically provided the perturbation is small enough that there is no need to control crossing of energy levels.

**Lemma.** *Let the dynamics $\tau_t$ be implemented by H. Consider a perturbation of the dynamics $\tau_t^{\lambda A}$ where $H \to H + \lambda A$. Consider a state $\omega$ whose corresponding density matrix is invertible. It is dynamically stable in the sense of Definition 1 if*

*(i) the state is invariant under $\tau_t$;*
*(ii) the corresponding density matrix $\rho$ expressed in energy eigenfunctions $|\phi_i\rangle$ with eigenvalues $H|\phi_i\rangle = h_i|\phi_i\rangle$ is given by*

$$\rho = \sum_i^n r_i|\phi_i\rangle\langle\phi_i|, \tag{1}$$

*where $r_i = r_j$ if $h_i = h_j$.*

Further, let the state be expressed by the density matrix $\rho = \mathrm{e}^{-M}$. Then in first order in $\lambda$ the density matrix of the perturbed state is given by

$$\rho_{\lambda A} = \mathrm{e}^{-M-\lambda B}, \tag{2}$$

where

$$B = \sum_{i\neq j} \frac{m_i - m_j}{h_i - h_j} a_{ij}|\phi_i\rangle\langle\phi_j| \tag{3}$$

with $\langle\phi_i|B|\phi_j\rangle = 0$ if $h_i = h_j$. The coefficients $m_i$ and $a_{ij}$ refer to $r_i = \mathrm{e}^{-m_i}$ and $A = \sum_{ij} a_{ij}|\phi_i\rangle\langle\phi_j|$.

**Proof.** Evidently $[\rho, H] = 0$ and therefore also $[M, H] = 0$. We write the perturbed density matrix in first order in $\lambda$ as $\rho_\lambda = \mathrm{e}^{-M-\lambda B}$. It has to commute with $H + \lambda A$ and therefore comparing the contributions in first order,

$$[A, \mathrm{e}^{-M}] = \left[H, \int_0^1 \mathrm{d}\gamma\, \mathrm{e}^{-\gamma M} B\, \mathrm{e}^{-(1-\gamma)M}\right] = \int_0^1 \mathrm{d}\gamma\, \mathrm{e}^{-\gamma M}[M, A]\, \mathrm{e}^{-(1-\gamma)M} \tag{4}$$

which reduces to $[M, A] = [H, B]$. Taking into account that unitary equivalence tells $r_i(\lambda) = r_i(0)$ this implies $\langle\phi_i|B|\phi_i\rangle = 0$. Otherwise, taking into account that $M$ and $H$ have common eigenvectors for these eigenvectors

$$(m_i - m_j)\langle\phi_i|A|\phi_j\rangle = (h_i - h_j)\langle\phi_i|B|\phi_j\rangle. \tag{5}$$

This $B$ is only well defined iff $M$ is degenerate in the degeneracy space of $H$ and therefore a function of $H$. □

## 3. Dynamical stability for the quasilocal algebra

The quasilocal algebra $\mathcal{A}$ (see [3] for a precise definition) represents the observable algebra in the thermodynamic limit. It consists of observables which are either located in finite regions or can be approximated in norm by such observables. Space translation acts as an automorphism $\sigma_x$ on it, either representing discrete $\sigma_x, x \in Z^d$, or continuous translations $\sigma_x, x \in R^d$, depending on whether the algebra represents a lattice system or a continuous system. Let us assume that time evolution corresponds to a strongly continuous automorphism on the algebra, i.e. with $A \in \mathcal{A}$ also $\tau_t(A) \in \mathcal{A}$ and $\lim_{t\to 0} \|\tau_t(A) - A\| = 0$. In [4] the following theorem is proven:

**Theorem 1.** *The following conditions are sufficient to guarantee that a state ω is a KMS state with respect to the time evolution $\tau_t$ for some inverse temperature β:*

*(1) the function $t \to \|[A, \tau_t(B)]\|$ is an $L^{(1)}$ function;*
*(2) ω is invariant under the automorphism $\tau_t, t \in R$;*
*(3)*

$$\lim_{t\to\infty} (\omega(A_1\tau_s(A_2)\tau_t(B_1)\tau_{t+s}(B_2)) - \omega(A_1\tau_s(A_2))\omega(B_1\tau_s(B_2)) = 0 \quad (6)$$

*uniformly in s;*
*(4) the state ω is dynamically stable in the sense that under a local perturbation of the dynamics to $\tau_t^{\lambda A}$ there exists a state $\omega_{\lambda A}$ with*

$$\|\omega - \omega_{\lambda A}\| \leqslant \lambda c_A, \qquad c_A \in R^+. \quad (7)$$

The proof first controls that $\int_{-\infty}^{\infty} \mathrm{d}t\,\omega([B, \tau_t(A)]) = 0\ \forall A, B$ and then by varying over $B = B_1\tau_s(B_2)$, $A = A_1\tau_s(A_2)$ shows that multitime correlations satisfy the KMS property. This is a strong result, but unfortunately the assumptions have been verified only for a few exceptional cases. Control of asymptotic Abelianess in models with interaction seems to be out of reach and even the existence of the time evolution as an automorphism was shown only for lattice systems [3] and with some manipulation on the interaction for continuous Fermi systems [10]. This last difficulty can be overcome: as is shown in [3], the dynamical stability can also be formulated if a time invariant state can be constructed and the time evolution exists as automorphism on the weak closure of the algebra in the corresponding GNS representation. Norm asymptotic Abelianess can be replaced by assuming $L^{(1)}$ properties of multitime correlation functions. However, this amounts to strong asymptotic Abelianess, i.e. it is only the topology that is weakened as it is now sufficient that

$$st - \lim_{t\to\infty} (\tau_t(\pi(A))\pi(B) - \pi(B)\tau_t(\pi(A))) = 0$$

fast enough. But again this property cannot be proven so far in explicit examples with interaction.

In this paper we will translate the considerations for the finite-dimensional algebra to the quasilocal algebra. There it was crucial for our construction of the perturbed state that we had control on the joined spectrum of the Hamiltonian and the density matrix implementing the state. Such control is available for the quasilocal algebra too; we have to replace only the density matrix by the modular operator and have to take into account that both the Hamiltonian and the modular operator do not belong to the algebra but act in the Hilbert space of the GNS representation. We collect the basic facts:

**Lemma.** *Let ω be a time invariant state. We consider the GNS representation where $\omega(A) = \langle\Omega|\pi(A)|\Omega\rangle$. Then the time evolution can be implemented unitarily and we can* *choose the unitary* $U_t = \mathrm{e}^{\mathrm{i}Ht}$ *such that* $U_t|\Omega\rangle = |\Omega\rangle$. *Now assume in addition that* $\omega$ *is modular, i.e. a state such that the GNS vector is separating.(This replaces the condition that* $\rho$ *is invertible for the finite case.) Then there exists the modular operator* $\Delta$ *which we write as* $\mathrm{e}^{-M}$ *and which is specified by the condition that*

$$\langle\Omega|\pi(A^\dagger)\pi(A)|\Omega\rangle = \langle\Omega|\pi(A)\,\mathrm{e}^{-M}\pi(A^\dagger)|\Omega\rangle \qquad \forall A \in \mathcal{A}. \tag{8}$$

It follows that

$$[\mathrm{e}^{\mathrm{i}Ht}, \mathrm{e}^{\mathrm{i}sM}] = 0 \qquad \forall s, t. \tag{9}$$

Further

$$H = -JHJ, \qquad M = -JMJ, \tag{10}$$

where $J$ is the modular conjugation corresponding to the modular operator $\mathrm{e}^{-M}$, i.e.

$$J\,\mathrm{e}^{-M/2}A|\Omega\rangle = A^*|\Omega\rangle \qquad \forall A \in \mathcal{A}. \tag{11}$$

**Lemma.** *Let* $\omega$ *be a state with separating GNS vector. Assume that* $\pi(\mathcal{A})''$ *is a factor. Let* $\omega$ *be invariant under* $\alpha_s$ *implemented by* $\mathrm{e}^{\mathrm{i}sP}$ *and also invariant under* $\sigma_x$ *where* $\sigma_x$ *is an automorphism that is strongly asymptotically Abelian and commutes with* $\alpha_s$. *Then the spectrum of P is additive, i.e. with* $a, b \in \mathrm{spec}(P)$*, also* $(a+b) \in \mathrm{spec}(P)$ *and as a consequence also* $na + mb, n, m \in N$.

The assumption that $\omega$ is a factor state means that $\pi(\mathcal{A})' \cap \pi(\mathcal{A})'' = c1$, i.e. that weak limits of the operators either are $c1$ or do not commute with some other operators of the algebra. If the algebra is a full matrix algebra as in the finite situation then every state is a factor state. For infinite systems this is not necessarily so, but the central decomposition provides us with a unique way to decompose into factor states. Especially extremal KMS states are factor states. Therefore the assumption is not a severe restriction. We need it because we have to exclude the possibility that the state is a linear superposition of KMS states with different temperature. The proof for the additivity of the spectrum is given in [8]. It is based on the idea that we compare the contribution with the spectrum of the vectors $A|\Omega\rangle$, $B|\Omega\rangle$ and $A\sigma_x(B)|\Omega\rangle$ for $x$ tending to $\infty$ and use the clustering of the state with respect to space translations. This clustering is guaranteed by the assumption, that our state is a factor state. We can apply this additivity both to $M$ and $H$ and as well to $M + cH$ showing that also the joined spectrum is additive. Thus, as a consequence of (10) we may take some $(m_1, h_1 > 0)$ and $(m_2, h_2 < 0)$ in the joined spectrum. Then we choose to every $\epsilon > 0$ natural numbers $n_1, n_2 \in \mathcal{N}$ such that $|n_1h_1 + n_2h_2| < \epsilon$. $|n_1m_1 + n_2m_2| < c\epsilon$ for all these numbers only if $H = \pm cM$. Especially this implies that $MH^{-1}$ is a bounded operator only if $H = \pm cM$ for some $c \in R$.

In [8] it was shown that for passive states, i.e. states that under a cyclic perturbation can only gain energy necessarily $MH \geqslant 0$ and additivity of the spectrum then implies $M = \beta H$ for some $\beta > 0$. Now we will see that for states that are dynamically stable $H^{-1}M(1-|\Omega\rangle\langle\Omega|)$ has to be a bounded operator, replacing (5) in the finite-dimensional situation, but, as we have just argued, this is in contradiction to the additivity of the spectrum if $H$ does not depend linearly on $M$. Equipped with these tools we can now weaken the condition (i) in theorem 1, and together with some technical assumptions we can prove the following theorem:

**Theorem 2.** *The following conditions are sufficient to guarantee that a state* $\omega$ *is a KMS state with respect to the time evolution* $\tau_t$ *for some inverse temperature:*

*(i)* *for* $A = A^* \in \pi_\omega(\mathcal{A})''$ *the condition* $\tau_t(A) = A$ *implies* $A = c_A 1, c_A \in R$, *i.e. the von Neumann algebra that is the weak closure of the GNS representation of the algebra corresponding to the state* $\omega$ *does not contain time invariant operators except multiples of unity;*

*(ii)* $\omega$ *is a factor state over the quasilocal algebra, invariant under space translation and time translation;*

*(iii)* *its GNS vector is separating;*

*(iv)* *it is dynamically stable in the sense that under all perturbations* $\tau_t^{\lambda A}$ *of the dynamics given by any bounded operator* $A = A^* \in \pi(\mathcal{A})''$ *there exists a perturbed state* $\omega_{\lambda A}$ *differentiable in norm with respect to* $\lambda$ *that is invariant under the perturbed dynamics.*

**Proof.** The assumption that the perturbed state is differentiable in norm implies that it can be implemented by some vector $|\Omega_{\lambda A}\rangle$ in the same Hilbert space with the same representation. Without any restriction we may choose the vector such that it belongs to the positive cone and therefore can be approximated in analogy to (2) in first order in $\lambda$, with some operator $B = B^* \in \pi(\mathcal{A})''$ as

$$|\Omega_{\lambda A}\rangle = \mathrm{e}^{\frac{-(M-\lambda B+J\lambda BJ)}{2}}|\Omega\rangle = |\Omega\rangle + \lambda\int_0^{\frac{1}{2}} \mathrm{d}\gamma\, \mathrm{e}^{-\gamma M} B|\Omega\rangle + O(\lambda^2). \tag{12}$$

Since we have chosen $|\Omega_{\lambda A}\rangle$ in the positive cone $|\Omega_{\lambda A}\rangle = J|\Omega_{\lambda A}\rangle$. Also the unitary $U_{t,\lambda A}$ that implements $\tau_t^{\lambda A}$ has to satisfy $U_{t,\lambda A} = JU_{t,\lambda A}J$ and therefore reads $\mathrm{e}^{\mathrm{i}t(H+\lambda A-J\lambda AJ)}$. Invariance of the perturbed state implies

$$0 = (H + \lambda A - \lambda JAJ)|\Omega_{\lambda A}\rangle. \tag{13}$$

Next we apply the connection between the modular conjugation and the modular automorphism given in (11) and observe

$$(A - JAJ)|\Omega\rangle = \left(A - \mathrm{e}^{-\frac{M}{2}}A\right)|\Omega\rangle = -\int_0^{\frac{1}{2}} \mathrm{d}\gamma\, \mathrm{e}^{-\gamma M} MA|\Omega\rangle.$$

With $H|\Omega\rangle = 0$ and using (9) we expand in $\lambda$ and obtain

$$\int_0^{\frac{1}{2}} \mathrm{d}\gamma\, \mathrm{e}^{-\gamma M} MA|\Omega\rangle = \int_0^{\frac{1}{2}} \mathrm{d}\gamma\, \mathrm{e}^{-\gamma M} HB|\Omega\rangle \tag{14}$$

which replaces (5). By assumption $|\Omega\rangle$ is cyclic and separating. Further $MA|\Omega\rangle = [MA - AM]|\Omega\rangle$ and similarly $HB|\Omega\rangle = [HB - BH]|\Omega\rangle$. Both $[HB - BH]$ and $[MA - AM]$ belong to the algebra; therefore the separability implies that we have to find a $B$ with $[M, A] = [H, B]$ in complete analogy to the finite-dimensional case (5). We demand that $\|\Omega_{\lambda A}\| = 1$, therefore $\langle\Omega|\frac{\mathrm{d}}{\mathrm{d}\lambda}\Omega_{\lambda A}\rangle|_{\lambda=0} = 0$ and consequently $\omega(B) = 0$. This fixes the ambiguity for $B$, since we assumed that there do not exist time invariant operators $B$.

It suffices that $B$ is affiliated to the algebra; there is no need that it is bounded, but $H^{-1}MA|\Omega\rangle$ has to be well defined for all operators $A$. Applying the additivity of the joined spectrum we can construct an $A$ such that $MA|\Omega\rangle$ is a well-defined vector that does not belong to the domain of $H^{-1}$, this construction only fails when $H^{-1}M = \beta$. Therefore only in this situation the assumption (*iv*) is met, $\omega$ has to be a temperature state to some temperature $\beta$. This finishes the proof. □

We want to compare the result with the finite case. The assumption that the only time invariant operators are multiples of the identity cannot be satisfied in finite-dimensional systems where the time evolution is implemented by an inner operator. For infinite systems it holds if the time evolution is G-Abelian [3], which includes strong asymptotic Abelianess but is a weaker condition. Especially for the $XY$-model this condition is satisfied. Further it can be shown that with some additional assumptions on spatial clustering it holds for Galilei invariant time evolution [12]. In the proof we need it as a replacement for the assumption—used in the finite-dimensional situation—that the perturbed density matrix should be unitarily equivalent to the initial one. The latter condition does not make sense for the infinite system where the state does not correspond to a density matrix. Note however that for the finite system we had to use a condition on the state, whereas for the infinite system we apply a condition on the dynamics which in the thermodynamic limit is satisfied for realistic models.

It seems worthwhile to compare this result with the other approaches. In [8] the result is stronger in the sense that negative temperatures are excluded. But the sign of the temperature is related to the direction of time, which plays no role in our description. That only positive temperatures are possible must be explained by other arguments such as stability of matter. Another important difference is the fact that in [8] it suffices to concentrate on a dense set of perturbations. All they have to do is to find an operator that contributes to the joined spectrum of $M$ and $H$ that lies in $R^+ \times R^-$. We have to find operators that do not belong to the intersection of the domains of $M$ and $H$. But this intersection is dense, at least in the von Neumann algebra. With $\gamma_s$ being the modular automorphism take the operator $A = \int \mathrm{d}t\, \mathrm{d}s f(t) g(s) \tau_t \gamma_s(C)$. Then the desired $B = \int \mathrm{d}t\, \mathrm{d}s F(t) g'(s) \tau_t \gamma_s(C)$ where $F(t) = \int_0^t \mathrm{d}s f(s)$. We only have to vary over all functions such that $f(t), g(s), F(t), g'(s)$ are $L^1$ functions and obtain a set of operators $A$ with $B|\Omega\rangle$ in the domain of $H^{-1}M$, norm dense, if $\gamma_s$ is not only an automorphism over $\pi(\mathcal{A})''$ but also over $\mathcal{A}$, otherwise only weakly dense. Note that also in the version of [4] it was sufficient to vary over a norm dense set of operators $A_i$ as long as they satisfy the assumption on the commutativity of $A_i \tau_s(A_j)$ uniformly in $s$. But in addition also cluster properties of the state had to be satisfied. In [3] they are expressed as

$$t \to \sup_{s\in R} |\omega(A_1 \tau_s(A_2) \tau_t(B_1 \tau_s(B_2))) - \omega(A_1 \tau_s(A_2)) \omega(B_1 \tau_s(B_2))|$$

is an $L^1$ function for all $A_1, A_2, B_1, B_2$ in a norm dense $\tau$ invariant subalgebra of $\mathcal{A}$. Therefore also in theorem 1, not only assumptions on the dynamics are important but also on the state and its cluster properties with respect to the dynamics, not only its reaction on perturbations of the dynamics.

It remains to wonder whether we get some deeper insight into the dynamics if we know which operators serve to test dynamical stability and how they are related to the state that is not dynamically stable. For quasifree time evolutions with continuous spectrum odd elements for instance can serve to destroy dynamical stability, but if we consider only the even algebra as relevant we have to look for other examples. Further investigations about the relation between the invariant state and the operator that destroys stability might give some deeper insight into the time evolution and time correlations. As an illustration we give the following example:

**Example.** We consider a quasifree time evolution with $\tau_t(a(f)) = a(\mathrm{e}^{\mathrm{i}ht})$ with $h(p)$ a multiplication operator in $p$-space. Let $\omega$ be a KMS state with respect to the automorphism $\gamma_s(a(f)) = a(\mathrm{e}^{\mathrm{i}mt} f)$ with $m(p)$ also a multiplication operator in $p$-space which for a lattice system is $[0, 1]$. Obviously $\tau$ and $\gamma$ commute. We perturb the dynamics with $A_1 = \int \mathrm{d}p\, \mathrm{d}q f(p, q) a_p^\dagger a_q$ where $f$ has to be a $L^2$ function. The corresponding $B_1$ reads $B_1 = \int \mathrm{d}p\, \mathrm{d}q \frac{m(p)-m(q)}{h(p)-h(q)} f(p, q) a_p^\dagger a_q$ and will be a bounded operator if $m(p), h(p)$ are smooth and monotonic functions. If however we take $A_2 = \int \mathrm{d}p_1\, \mathrm{d}p_2\, \mathrm{d}q_1\, \mathrm{d}q_2 f(p_1, p_2, q_1, q_2) a_{p_1}^\dagger a_{p_2}^\dagger a_{q_1} a_{q_2}$ then

$$B_2 = \int \mathrm{d}p_1\, \mathrm{d}p_2\, \mathrm{d}q_1\, \mathrm{d}q_2 \frac{m(p_1) + m(p_2) - m(q_1) - m(q_2)}{h(p_1) + h(p_2) - h(q_1) - h(q_1)} f(p_1, p_2, q_1, q_2) a_{p_1}^\dagger a_{p_2}^\dagger a_{q_1} a_{q_2}.$$

Now, even if $m(p)$ and $h(p)$ are smooth functions the denominator can vanish whereas the nominator does not vanish; therefore already with an appropriate choice for $f$, the operator $A_2$ is a perturbation for which the invariant state is not stable.

Note that similar considerations occur in the framework of nonequilibrium steady states (NESS) [13]. Here perturbations of invariant states are considered, especially of tensor product states with different temperatures. On the basis of a scattering mechanism which is continuous in the perturbation parameter, with some assumptions on the perturbations, such a state is driven to a state invariant under the perturbed dynamics which is not normal with respect to the original state. Therefore in the spirit of our investigation these perturbations serve as test for dynamical stability.

## Appendix

It remains to justify why it was desirable to extend the result of [4] to dynamics that are not norm asymptotic Abelian by offering a concrete example that violates this assumption: we consider the $XY$-model in its simplest version, variations in the parameters will not change the main considerations. The Hamiltonian reads

$$H = \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + cS_i^z\right). \tag{A.1}$$

Here $S_i$ are the Pauli matrices at position $i$. If we consider a finite system where the sum runs over $0 \leqslant i \leqslant N$ we can apply a Jordan–Wigner transformation such that the Hamiltonian can be expressed in creation and annihilation operators

$$H = \sum_i \left(a_{i+1}^\dagger a_i + a_i^\dagger a_{i+1}\right)$$

up to some boundary terms. All operators that are even products in $S^x$, $S^y$ and contain an arbitrary but finite number of $S^z$ can be expressed in terms of creation and annihilation operators with only a finite number of contributions from different lattice points, independent of $N$. Therefore, as is discussed in detail in [6] and where it is also justified that we may ignore the boundary terms, all these operators inherit the quasifree time evolution which is well under control. For our choice of parameters this quasifree time evolution corresponds on the one-particle level to a Hamiltonian with a continuous spectrum. Therefore the time evolution is norm asymptotically Abelian on the even algebra which is the same for the creation and annihilation operators as for the $XY$-model [11].

It remains to transfer the time evolution to the odd polynomials in the $XY$-model. It is sufficient if we can control $\tau_t\left(S_0^x\right)$ since all other odd elements can be expressed as a product of $S_0^x$ with an even element on which the evolution is given. $S_0^x$ defines an automorphism $\alpha$ on the even elements

$$\alpha(A) = S_0^x A S_0^x. \tag{A.2}$$

Therefore

$$\alpha\tau_t\alpha\tau_{-t}(A) = S_0^x\tau_t\left(S_0^x\right)A\tau_t\left(S_0^x\right)S_0^x = V_t A V_t^\dagger. \tag{A.3}$$

For finite times and finite $N\,V_t(N)$ is again an even element and can be written as

$$S_0^x\,\mathrm{e}^{\mathrm{i}H_N t}S_0^x\,\mathrm{e}^{-\mathrm{i}Ht} = \mathrm{e}^{\mathrm{i}\alpha H_N t}\,\mathrm{e}^{-\mathrm{i}H_N t} = \mathrm{e}^{\mathrm{i}H_N t + iBt}\,\mathrm{e}^{-\mathrm{i}H_N t}.$$

$B$ is a local and even operator, namely for our choice of the Hamiltonian $B = 2\left(S_0^y S_1^y + S_0^y S_{-1}^y + S_0^z\right)$ and is therefore quadratic in creation and annihilation operators. Therefore

$$\lim_{N\to\infty} V_t(N) = V_t \tag{A.4}$$

exists and is an even element and for our choice of the Hamiltonian again quadratic in creation and annihilation operators. We are interested in the asymptotic commutation relations, i.e. whether

$$\lim_{t\to\infty}\left[S_0^x, \tau_t S_0^x\right] = \lim_{t\to\infty}\left(V_t - V_t^\dagger\right) \neq 0. \tag{A.5}$$

We can consider

$$\lim_{t\to\infty} V_t A V_t^\dagger. \tag{A.6}$$

If we choose for $A$ a creation or annihilation operator we can calculate (20) on the one-particle level and can apply scattering theory. The corresponding operator on the one-particle level exists and therefore also the corresponding scattering automorphism on the algebra. However the scattering operator on the one-particle level does not satisfy the necessary Hilbert–Schmidt properties [14] so that the automorphism is not an inner automorphism and therefore $\lim_{t\to\infty} V_t$ does not exist. In addition we have to make sure that $\lim_{t\to\infty} V_t V_t^\dagger \neq 1$ does not exist either. But this would correspond to a trivial scattering operator on the one-particle level which does not hold.

It follows that on the odd elements the time evolution of the $XY$-model is not norm asymptotic Abelian. In addition it guarantees that no time invariant odd element exists. On the even algebra norm asymptotic Abelianess already excluded the existence of time invariant operators other than multiplicity of unity; therefore though the assumptions of theorem 1 are violated, the assumptions of theorem 2 are met.

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